Rates in almost sure invariance principle for slowly mixing dynamical systems

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joint work with C.Cuny, J.Dedecker and A. Korepanov

Long-Time Behaviour and Statistical Inference for Stochastic Processes: from Markovian to Long-Memory Dynamics.
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Let \((X_i)_{i \geq 1}\) be a sequence of real-valued r.v.’s centered and with second moment \(\sigma^2\) finite. Let \(S_n = X_1 + X_2 + \cdots + X_n\).
Strong approximations in the iid setting

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- The almost sure invariance principle (ASIP) states that one can redefine \((S_n)_{n \geq 1}\) without changing its distribution on a (richer) probability space on which there exists a sequence \((Z_i)_{i \geq 1}\) of iid centered Gaussian variables with variance \(\sigma^2\) such that

\[
\max_{k \leq n} |S_k - \sigma B_k| = o(b_n) \text{ almost surely,}
\]

where \(b_n = (n \log \log n)^{1/2}\) (Strassen (1964)) and \(B_k = \sum_{i=1}^{k} Z_i\).
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When \((X_i)_{i\geq 1}\) is assumed to be in addition in \(L^p\) with \(p > 2\), then we can obtain rates in the ASIP:

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b_n = n^{1/p}
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(see Major (1976) for \(p \in ]2, 3]\) and Komlós-Major-Tusnády (1975) for \(p > 3\).
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- These results are based on a recursive dyadic construction using the conditional quantile method (this method is called the ”Hungarian construction”).
And what about if the r.v.’s do not have the same law?

- **Sakhanenko (06’).** Let \((X_i)_{i \geq 1}\) be a sequence of independent r.v.’s centered and in \(L^2\). Let \(r > 2\). On a richer probability space, one can construct a sequence \((Z_i)_{i \geq 1}\) of independent centered gaussian r.v.’s with \(\text{Var}(Z_n) = \text{Var}(X_n)\) and such that for all \(x > 0\) and all \(n \geq 1\),

\[
\mathbb{P} \left( \max_{1 \leq k \leq n} |S_k - B_k| > c(r)x \right) \leq \sum_{i=1}^{n} \mathbb{E} \min \left( \frac{|X_i|^r}{x^r}, \frac{|X_i|^2}{x^2} \right).
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In the iid setting and in the one-dimensional case, the rate in the ASIP is \(O(\log n)\) as soon as the r.v.’s have a finite moment generating function in a neighborhood of 0 (KMT, '76). This rate is unimprovable!
Some extensions in the Dependent setting

- In 2014, Berkes-Liu-Wu proved the ASIP with rate $o(n^{1/p})$, $p > 2$, when $X_k = g(\ldots, \varepsilon_{k-1}, \varepsilon_k)$ with $(\varepsilon_k)_{k \in \mathbb{Z}}$ are iid r.v.'s, $\|X_0\|_p < \infty$ and assuming some weak dependence conditions.
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- They assume an arithmetical decay of convergence of $\|X_k - X_k^*\|_p$ where $X_k^* = g(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)$.
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- In 2018, Cuny-Dedecker, M. used the idea developed in BLW to obtain sharp conditions for the ASIP with rate \( o(n^{1/p}) \) in case of functions of random iterates. They considered models where \( X_n = h(\varepsilon_n, W_{n-1}) \) with \( W_n = F(\varepsilon_n, W_{n-1}) \).
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- For all these works, the fact that there is an underlying sequence of iid r.v.’s and the representation by this sequence is known plays a crucial role.
Rates in the ASIP for some slowly dynamical systems

Our toy model: let us consider the LSV map (Liverani, Saussol et Vaienti, 1999):

for $0 < \gamma < 1$, \( f(x) = \begin{cases} 
  x(1 + 2\gamma x) & \text{if } x \in [0, 1/2[ \\
  2x - 1 & \text{if } x \in [1/2, 1] 
\end{cases} \)

Graph of \( f \)
There exists a unique absolutely continuous $f$-invariant probability measure $\mu$ on $[0, 1]$, which is equivalent to the Lebesgue measure and whose density $h$ satisfies $0 < c \leq x^\gamma h(x) \leq C < \infty$. 

The intermittent behaviour comes from the fact that $0$ is a fixed point with $f'(0) = 1$. Hence if a point $x$ is close to $0$, then its orbit $(f^n(x))_{n \geq 0}$ stays around $0$ for a long time. The degree of intermittency is given by the parameter $\gamma$ and is quantified by choosing an interval away from $0$ such as $Y = \frac{1}{2}, 1$ and considering the first return time $\tau$: 

\[ \tau(x) = \min\{n \geq 1 : f^n(x) \in Y\} \]

We have 

\[ C^{-1} n^{-1/\gamma} \leq \text{Leb}(\tau \geq n) \leq C n^{-1/\gamma} \quad (\text{Gouezel'04 or Young'99}) \]

For this model, the return time has a weak moment of order $\beta = 1/\gamma$. 

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We have $C^{-1} n^{-1/\gamma} \leq \text{Leb}(\tau \geq n) \leq Cn^{-1/\gamma}$ (Gouezel’04 or Young’99)

For this model, the return time has a weak moment of order $\beta = 1/\gamma$. 
Suppose that $\varphi: [0, 1] \to \mathbb{R}$ is a Hölder continuous observable with $\int \varphi \, d\mu = 0$ and let

$$S_n(\varphi) = \sum_{k=0}^{n-1} \varphi \circ f^k.$$
Suppose that $\varphi : [0, 1] \to \mathbb{R}$ is a Hölder continuous observable with $\int \varphi \, d\mu = 0$ and let

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We consider $S_n(\varphi)$ as a discrete time random process on the probability space $([0, 1], \mu)$. The increments $(\varphi \circ f^n)_{n \geq 0}$ are stationary (since $\mu$ is $f$-invariant) and their covariances decay polynomially:

$$\left| \int \varphi \, \varphi \circ f^n \, d\mu \right| = O\left(n^{-(1-\gamma)/\gamma}\right).$$
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If $\gamma < 1/2$, $n^{-1/2} S_n(\varphi) \to^d N(0, c^2)$ with

$$c^2 = \int \varphi^2 \, d\mu + 2 \sum_{n=1}^{\infty} \int \varphi \, \varphi \circ f^n \, d\mu \quad (\ast)$$
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What about rates in the ASIP when $\gamma < 1/2$?
Previous results for the LSV map

- The first results were obtained by Melbourne and Nicol (2005) but without explicit rates by using a coupling method due to Philipp and Stout (1975).

Using an approximation via reverse martingale difference sequences and an ASIP for reverse MDS due to Cuny-M. ('15), Korepanov-Kosloff-Melbourne '16 proved the ASIP with rates $S_n(\phi) - W_n = o\left(n^{\gamma} + \epsilon\right)$, $\gamma \in \left[\frac{1}{4}, \frac{1}{2}\right]$, for all $\epsilon > 0$. 

For Hölder continuous or bounded variation observables, using a conditional quantile method, M.-Rio '12, proved the ASIP with rates $S_n(\phi) - W_n = O\left(n^{\gamma'}(\log n)^{1/2}(\log \log n)^{1/4}(1 + \epsilon)^{\gamma'}\right)$ for all $\epsilon > 0$, where $\gamma' = \max\{\gamma, 1/3\}$.

Is it possible to get better rates than $O\left(n^{1/4}\right)$? For $\phi = Id$ and $f(x) = 2x \mod 1$, one can have much better!
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\[ S_n(\varphi) - W_n = \begin{cases} 
    o(n^{\gamma+\varepsilon}), & \gamma \in [1/4, 1/2[ \\
    O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}), & \gamma \in ]0, 1/4[ 
\end{cases} 
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for all \( \varepsilon > 0 \) (No way to get better bounds with this method!)
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- Is it possible to get better rates than \( O(n^{1/4}) \)? For \( \varphi = Id \) and \( f(x) = 2x \ mod 1 \), one can have much better!
Our results

Theorem (Cuny-Dedecker-Korepanov-M. (2019))

Let $\gamma \in (0, 1/2)$ and $\varphi : [0, 1] \to \mathbb{R}$ be a Hölder continuous observable with $\int \varphi \, d\mu = 0$. For the LSV map, the random process $S_n(\varphi)$ satisfies the ASIP with variance $c^2$ given by (*) and rate $o(n^{\gamma}(\log n)^{\gamma+\epsilon})$ for all $\epsilon > 0$. If $c^2 = 0$, the rate in the ASIP can be improved to $O(1)$. Indeed, in this case $\varphi$ is a coboundary in the sense that $\varphi = u - u \circ f$ with $u$ bounded. However in general the rates are optimal in the following sense:

Proposition (C-D-K-M. (2019))

There exists a Hölder continuous observable $\varphi$ with $\int \varphi \, d\mu = 0$ such that $\limsup_{n \to \infty} \left( n^{\gamma}(\log n)^{\gamma+\epsilon} \right) > 0$ for all Brownian motions $(W_t)_{t \geq 0}$ defined on the same (possibly enlarged) probability space as $(S_n(\varphi))_{n \geq 0}$. 
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If \( c^2 = 0 \), the rate in the ASIP can be improved to \( O(1) \). Indeed, in this case \( \varphi \) is a *coboundary* in the sense that \( \varphi = u - u \circ f \) with \( u \) bounded.

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**Proposition (C-D-K-M. (2019))**

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\limsup_{n \to \infty} (n \log n)^{-\gamma} |S_n(\varphi) - W_n| > 0
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for all Brownian motions \((W_t)_{t \geq 0}\) defined on the same (possibly enlarged) probability space as \((\tilde{S}_n(\varphi))_{n \geq 0}\).
Another toy model: The Holland map ('05)

For a $\gamma \in (0, 1)$, let

$$f(x) = \begin{cases} 
x(1 + x^\gamma \rho(x)), & x \leq 1/2 \\
2x - 1, & x > 1/2 
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where $\rho(x) = C |\log x|^{(1+\varepsilon)\gamma}$ with $\varepsilon > 0$ and $C = 2^\gamma (\log 2)^{-(1+\varepsilon)\gamma}$. 

The first return time has a strong moment of order $\beta = 1/\gamma$:

$$\int \tau^{1/\gamma} d\text{Leb} < \infty.$$ 

In this situation we have

Theorem (Cuny-Dedecker-Korepanov-M. (2019))

Let $\gamma \in (0, 1/2)$ and $\phi: [0, 1] \to \mathbb{R}$ be a Hölder continuous observable with $\int \phi d\mu = 0$. For the Holland map, the random process $S_n(\phi)$ satisfies the ASIP with variance $c^2$ given by (*) and rate $o(n^\gamma)$.

Intermittent maps are prototypical examples of nonuniformly expanding dynamical systems. Similar results can be obtained in this general setup (the rates depend on the weak or strong moments of the return time).
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  $$f(x) = \begin{cases} x(1 + x^\gamma \rho(x)), & x \leq 1/2 \\ 2x - 1, & x > 1/2 \end{cases}$$

  where $\rho(x) = C |\log x|^{(1+\varepsilon)\gamma}$ with $\varepsilon > 0$ and $C = 2^\gamma (\log 2)^{-(1+\varepsilon)\gamma}$.

- The first return time has a strong moment of order $\beta = 1/\gamma$:
  $$\int_Y \tau^{1/\gamma} \, d\text{Leb} < \infty.$$ 

- In this situation we have

**Theorem (Cuny-Dedecker-Korepanov-M. (2019))**

*Let $\gamma \in (0, 1/2)$ and $\varphi: [0, 1] \to \mathbb{R}$ be a Hölder continuous observable with $\int \varphi \, d\mu = 0$. For the Holland map, the random process $S_n(\varphi)$ satisfies the ASIP with variance $c^2$ given by (*) and rate $o(n^\gamma)$.*

- Intermittent maps are prototypical examples of *nonuniformly expanding dynamical systems*. Similar results can be obtained in this general setup (the rates depend on the weak or strong moments of the return time).
Main ideas of the proof

The main idea is to construct a stationary Markov chain $(g_n, n \in \mathbb{N})$ on a countable space $S$ and an observable $\psi : \Omega \to \mathbb{R}$ (here $\Omega \subset S^\mathbb{N}$) such that $\int \psi \, d\mathbb{P}_\Omega = 0$ and setting

$$X_k = \psi(g_k, g_{k+1}, \ldots), \ k \geq 0,$$

the process $(X_k)_{k\geq0}$ on the probability space $(\Omega, \mathbb{P}_\Omega)$ is equal in law to $(\varphi \circ f^k)_{k\geq0}$ on $([0, 1], \mu)$. 

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- Our Markov chain is in the spirit of the classical Young towers.

- Recall that $Y = ]1/2, 1]$ and $\tau : Y \to \mathbb{N}$ be the inducing time $\tau(x) = \min\{n \geq 1 : f^n(x) \in Y\}$. Let $F : Y \to Y$ be the induced map: $F(x) = f^{\tau(x)}(x)$. Let $\alpha$ be the partition of $Y$ into the intervals where $\tau$ is constant.
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- Our Markov chain is in the spirit of the classical Young towers.

- Recall that \(Y =]1/2, 1]\) and \(\tau : Y \to \mathbb{N}\) be the inducing time \(\tau(x) = \min\{n \geq 1 : f^n(x) \in Y\}\). Let \(F : Y \to Y\) be the induced map: \(F(x) = f^{\tau(x)}(x)\). Let \(\alpha\) be the partition of \(Y\) into the intervals where \(\tau\) is constant.

- Let \(A\) denote the set of all finite words in the alphabet \(\alpha\), not including the empty word. Denote by \(w = a_0 \cdots a_{n-1}\) an element of \(A\). Let also \(h : A \to \mathbb{N}\), \(h(w) = \tau(a_0) + \cdots + \tau(a_{n-1})\).
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Let $g_0 \in S$ be distributed according to a certain $\nu$ and $(\varepsilon_k)$ be a sequence of iid r.v. with values in $A$, distribution $\mathbb{P}_A$ and independent from $g_0$. For the LSV and the Holland maps the constructed Markov chain is aperiodic.
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Then, for any $n \geq 0$, we define

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For the LSV and the Holland maps the constructed Markov chain is aperiodic.
We shall prove an ASIP for the partial sums associated with $\left( X_k \right)_{k \geq 0}$ where $X_k = \psi(g_k, g_{k+1}, \ldots)$. The function $\psi$ satisfies the following property: for $a = (g_0, \ldots, g_N, g_{N+1}, \ldots)$ and $b = (g_0, \ldots, g_N, g_{N+1}', \ldots)$ with $g_{N+1} \neq g_{N+1}'$, $|\psi(a) - \psi(b)| \leq C \theta \sum_{k=0}^{1} \{ g_k \in S_0 \}$, where $S_0 = \{ (w, 0) : w \in A \}$ and $\theta \in [0, 1]$. With the help of the above property one can prove that there exists a measurable function $G_m$ such that, for any $r \geq 1$, $\|X_k - G_m(\varepsilon_k - m, \ldots, \varepsilon_k + m)\|_1 \ll P(T \geq m) + m - r/2$ where $T$ is the meeting time $T = \inf \{ n \geq 0 : g_n = g_\ast n \}$ here $g_\ast 0$ has distribution $\nu$ and is independent of $(g_0, (\varepsilon_k)_{k \geq 1})$ and $g_\ast n + 1 = U(g_\ast n, \varepsilon_{n+1})$.
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a \(=(g_0, \ldots, g_N, g_{N+1}, \ldots)\) and \(b=(g_0, \ldots, g_N, g'_{N+1}, \ldots)\) with 
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\[|\psi(a) - \psi(b)| \leq C \theta^{\sum_{k=0}^{N} 1_{\{g_k \in S_0\}}},\]

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With the help of the above property one can prove that there exists 
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\[
\|X_k - G_m(\varepsilon_{k-m}, \ldots, \varepsilon_{k+m})\|_1 \ll \mathbb{P}(T \geq m) + m^{-r/2}
\]

where \(T\) is the meeting time

\[
T = \inf\{n \geq 0 : g_n = g^*_n\}
\]

here \(g^*_0\) has distribution \(\nu\) and is independent of \((g_0, (\varepsilon_k)_{k \geq 1})\) and \(g^*_{n+1} = U(g^*_n, \varepsilon_{n+1})\).
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The $2m$-dependent approximation

$$\|X_k - G_m(\varepsilon_{k-m}, \ldots, \varepsilon_{k+m})\|_1 \ll \mathbb{P}(T \geq m) + m^{-r/2}$$

allows to adapt the scheme of proof developed by Berkes-Liu-Wu ('14) to prove KMT with rate $o(n^{1/p})$ for functions of iid having a moment of order $p$, under a weak dependence condition.
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The $2m$-dependent approximation

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allows to adapt the scheme of proof developed by Berkes-Liu-Wu ('14) to prove KMT with rate $o(n^{1/p})$ for functions of iid having a moment of order $p$, under a weak dependence condition.

Their proof consists first of providing a conditional Gaussian approximation by freezing some part of the $(\varepsilon_k)_k$, making suitable blocks and applying Sakhanenko’s '06 result, and after of proceeding to a unconditional Gaussian approximation.
Some references


Thank you for your attention!