Rates in almost sure invariance principle for slowly mixing dynamical systems

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Strong approximations in the iid setting

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- The almost sure invariance principle (ASIP) states that one can redefine $(S_n)_{n\geq 1}$ without changing its distribution on a (richer) probability space on which there exists a sequence $(Z_i)_{i\geq 1}$ of iid centered Gaussian variables with variance σ^2 such that

$$\max_{k\leq n} |S_k - \sigma B_k| = o(b_n) \text{ almost surely,}$$

where $b_n = (n \log\log n)^{1/2}$ (Strassen (1964)) and $B_k = \sum_{i=1}^{k} Z_i$.

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- When $(X_i)_{i\geq 1}$ is assumed to be in addition in $\mathbf{L}^p$ with $p > 2$, then we can obtain rates in the ASIP:

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- These results are based on a recursive dyadic construction using the conditional quantile method (this method is called the "Hungarian construction").

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- Extensions in the multivariate setting were obtained by Einmahl ('87,'89), Zaitsev ('98, '07)
- In the iid setting and in the one-dimensional case, the rate in the ASIP is $O(\log n)$ as soon as the r.v.'s have a finite moment generating function in a neighborhood of 0 (KMT, '76). This rate is unimprovable !

Some extensions in the Dependent setting

- In 2014, Berkes-Liu-Wu proved the ASIP with rate $o(n^{1/p})$, $p > 2$, when $X_k = g(\ldots, \varepsilon_{k-1}, \varepsilon_k)$ with $(\varepsilon_k)_{k\in\mathbb{Z}}$ are iid r.v.' s, $\|X_0\|_p < \infty$ and assuming some weak dependence conditions.

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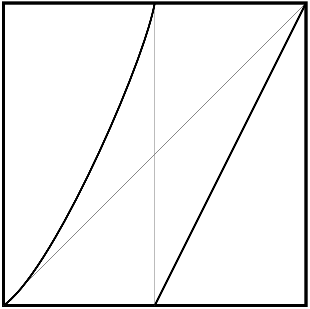
- For all these works, the fact that there is an underlying sequence of iid r.v.'s and the representation by this sequence is known plays a crucial role.

Rates in the ASIP for some slowly dynamical systems

Our toy model: let us consider the LSV map (Liverani, Saussol et Vaienti, 1999):

$$\text{for } 0 < \gamma < 1, \quad f(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$



Graph of f

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- The degree of intermittency is given by the parameter γ and is quantified by choosing an interval away from 0 such as $Y =]1/2, 1]$ and considering the first return time $\tau \colon Y \to \mathbb{N}$,

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- We have $C^{-1} n^{-1/\gamma} \leq \mathrm{Leb}\,(\tau \geq n) \leq C n^{-1/\gamma}$ (Gouezel'04 or Young'99)
- For this model, the return time has a weak moment of order $\beta = 1/\gamma$.

- Suppose that $\varphi\colon [0,1] \to \mathbb{R}$ is a Hölder continuous observable with $\int \varphi \, d\mu = 0$ and let

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- We consider $S_n(\varphi)$ as a discrete time random process on the probability space $([0,1], \mu)$. The increments $(\varphi \circ f^n)_{n \geq 0}$ are stationary (since μ is f-invariant) and their covariances decay polynomially:

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- If $\gamma < 1/2$, $n^{-1/2} S_n(\varphi) \to^d N(0, c^2)$ with

$$c^2 = \int \varphi^2 \, d\mu + 2 \sum_{n=1}^{\infty} \int \varphi \, \varphi \circ f^n \, d\mu \qquad (*)$$

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- What about rates in the ASIP when $\gamma < 1/2$?

Previous results for the LSV map

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$$S_n(\varphi) - W_n = \begin{cases} o(n^{\gamma+\varepsilon}), & \gamma \in [1/4, 1/2[\\ O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}), & \gamma \in]0, 1/4[\end{cases}$$

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- For Hölder continuous or bounded variation observables, using a conditional quantile method, M.-Rio '12, proved the ASIP with rates

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  for all $\varepsilon > 0$, where $\gamma' = \max\{\gamma, 1/3\}$.
- Is it possible to get better rates than $O(n^{1/4})$? For $\varphi = Id$ and $f(x) = 2x \; mod 1$, one can have much better !

Our results

Theorem (Cuny-Dedecker-Korepanov-M. (2019))

Let $\gamma \in (0, 1/2)$ and $\varphi\colon [0,1] \to \mathbb{R}$ be a Hölder continuous observable with $\int \varphi \, d\mu = 0$. For the LSV map, the random process $S_n(\varphi)$ satisfies the ASIP with variance c^2 given by (*) and rate $o(n^\gamma (\log n)^{\gamma+\varepsilon})$ for all $\varepsilon > 0$.

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Proposition (C-D-K-M. (2019))

There exists a Hölder continuous observable φ with $\int \varphi \, d\mu = 0$ such that

$$\limsup_{n\to\infty} (n \log n)^{-\gamma} |S_n(\varphi) - W_n| > 0$$

for all Brownian motions $(W_t)_{t\geq 0}$ defined on the same (possibly enlarged) probability space as $(S_n(\varphi))_{n\geq 0}$.

Another toy model: The Holland map ('05)

- For a $\gamma \in (0,1)$, let

$$f(x) = \begin{cases} x(1 + x^\gamma \rho(x)), & x \le 1/2 \\ 2x - 1, & x > 1/2 \end{cases}$$

where $\rho(x) = C|\log x|^{(1+\varepsilon)\gamma}$ with $\varepsilon > 0$ and $C = 2^\gamma (\log 2)^{-(1+\varepsilon)\gamma}$.

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- Intermittent maps are prototypical examples of *nonuniformly expanding dynamical systems*. Similar results can be obtained in this general setup (the rates depend on the weak or strong moments of the return time).

Main ideas of the proof

- The main idea is to construct a stationary Markov chain $(g_n, n \in \mathbb{N})$ on a countable space S and an observable $\psi : \Omega \to \mathbb{R}$ (here $\Omega \subset S^{\mathbb{N}}$) such that $\int \psi \, d\mathbb{P}_\Omega = 0$ and setting

$$X_k = \psi(g_k, g_{k+1}, \ldots) \, , \; k \geq 0,$$

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- Our Markov chain is in the spirit of the classical Young towers.
- Recall that $Y =]1/2, 1]$ and $\tau \colon Y \to \mathbb{N}$ be the inducing time $\tau(x) = \min\{n \geq 1 \colon f^n(x) \in Y\}$. Let $F : Y \to Y$ be the induced map: $F(x) = f^{\tau(x)}(x)$. Let α be the partition of Y into the intervals where τ is constant.

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  the process $(X_k)_{k \geq 0}$ on the probability space $(\Omega, \mathbb{P}_\Omega)$ is equal in law to $(\varphi \circ f^k)_{k \geq 0}$ on $([0,1], \mu)$.
- Our Markov chain is in the spirit of the classical Young towers.
- Recall that $Y =]1/2, 1]$ and $\tau \colon Y \to \mathbb{N}$ be the inducing time $\tau(x) = \min\{n \geq 1 \colon f^n(x) \in Y\}$. Let $F : Y \to Y$ be the induced map: $F(x) = f^{\tau(x)}(x)$. Let α be the partition of Y into the intervals where τ is constant.
- Let $\mathcal{A}$ denote the set of all finite words in the alphabet α, not including the empty word. Denote by $w = a_0 \cdots a_{n-1}$ an element of $\mathcal{A}$. Let also $h \colon \mathcal{A} \to \mathbb{N}$, $h(w) = \tau(a_0) + \cdots + \tau(a_{n-1})$

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- For the LSV and the Holland maps the constructed Markov chain is aperiodic.

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$$|\psi(a) - \psi(b)| \leq C\theta^{\sum_{k=0}^{N} \mathbf{1}_{\{g_k \in S_0\}}} ,$$

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- With the help of the above property one can prove that there exists a measurable function G_m such that, for any $r \geq 1$,

$$\|X_k - G_m(\varepsilon_{k-m}, \dots, \varepsilon_{k+m})\|_1 \ll \mathbb{P}(T \geq m) + m^{-r/2}$$

where T is the meeting time

$$T = \inf\{n \geq 0 : g_n = g_n^*\}$$

here g_0^* has distribution ν and is independent of $(g_0, (\varepsilon_k)_{k\geq 1})$ and $g_{n+1}^* = U(g_n^*, \varepsilon_{n+1})$.

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- Their proof consists first of providing a conditional Gaussian approximation by freezing some part of the $(\varepsilon_k)_k$, making suitable blocks and applying Sakhanenko's '06 result, and after of proceeding to a unconditional Gaussian approximation.

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Thank you for your attention!