Pathwise techniques for rough differential equations: existence and longtime behavior

María Garrido-Atienza (University of Sevilla, Spain)

Joint work with

Hongjun Gao (Nanjin Normal University, China)
Kening Lu (BYU, USA)
David Nualart (University of Kansas, USA)
Björn Schmalfuß (University of Jena, Germany)

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OUTLINE

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2. Solving differential equations with fBm by using Fractional calculus:

- **Case** $H \in (1/2, 1)$**.**

- **Case** $H \in (1/3, 1/2]$. **Exponential stability of the trivial solution.**

3. Solving differential equations with fBm $H \in (1/3, 1/2]$ **by using Rough paths.**

Motivation and Preliminaires

Goal: To study the existence and uniqueness of pathwise solutions as well as the random dynamical system generated by

$$du = Audt + G(u)d\omega, \qquad u(0) = u_0 \in V.$$

- **Assume that A generates a C_0–semigroup, G Lipschitz-continuous, ω Brownian motion.**

  Da Prato & Zabczyk (1992): For any $u_0 \in V$ there exists a mild solution almost surely which is continuous, adapted and unique modulo $\mathbb{P}$.

  Does this SPDE generate an RDS? Special answers

$$G(u)d\omega = d\omega, \qquad G(u)d\omega = u\,d\omega$$

- **ω is a Brownian motion: exceptional sets contradict the definition of an RDS.**

  Problem of infinite dimensional stochastic flows: Kolmogorov Test does not make sense for SPDE!

- **ω is a fractional Brownian motion (fBm): Pathwise interpretation of the integral with respect to the fBm.**

Given $H \in (0,1)$, a continuous centered Gaussian process $\beta^H(t)$, $t \in \mathbb{R}$, with

$$\mathbb{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t,\, s \in \mathbb{R}$$

is a one-dimensional fractional Brownian motion, and H is the Hurst parameter.

Let $(V, |\cdot|)$ be a separable Hilbert space, Q is a bounded and symmetric linear operator on V of trace class, i.e. for $(e_i)_{i\in\mathbb{N}}$ an orthonormal basis in V there exists a sequence of nonnegative numbers $(q_i)_{i\in\mathbb{N}}$ such that $trQ = \sum_{i=1}^{\infty} q_i < \infty$.

$$B^H(t) = \sum_{i=1}^{\infty} \sqrt{q_i} e_i \beta_i^H(t), \qquad t \in \mathbb{R},$$

is a continuous V-valued fBm B^H with Hurst parameter H, where $(\beta_i^H)_{i\in\mathbb{N}}$ is a sequence of stochastically one-dimensional fBm.
Lemma. The quadruple $(C_0(\mathbb{R}, V), \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, where θ_t is the Wiener shift:

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R},\ \omega \in \Omega$$

is an ergodic metric dynamical system. We identify $B^H(\cdot, \omega)$ and $\omega(\cdot)$.

Moreover, there exists a version with β'–Hölder continuous paths for any $\beta' < H$.

Fractional Calculus: case $H \in (1/2, 1)$.

For $f \in C^{\beta}([0,T];V)$**,** $\omega \in C^{\beta'}([0,T];V)$**,** $\alpha < \beta$**,** $1-\alpha < \beta'$ **we interpret the stochastic integral as the generalized Stieltjes integral**

$$\int_0^T f d\omega = (-1)^{\alpha} \int_0^T D_{0+}^{\alpha} f(r) D_{T-}^{1-\alpha} \omega_{T-}(r) dr,$$

where $\omega_{T-}(r) := \omega(r) - \omega(T)$ **and**

$$D_{0+}^{\alpha} f(r) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(r)}{r^{\alpha}} + \alpha \int_0^r \frac{f(r)-f(q)}{(r-q)^{\alpha+1}} dq \right),$$

$$D_{T-}^{1-\alpha} \omega_{T-}(r) = \frac{(-1)^{\alpha}}{\Gamma(\alpha)} \left(\frac{\omega(r)-\omega(T)}{(T-r)^{1-\alpha}} + (1-\alpha) \int_r^T \frac{\omega(r)-\omega(q)}{(q-r)^{2-\alpha}} dq \right).$$

- **Samko, Kilbas and Marichev (1993): exhaustive survey on classical fractional calculus.**

- **This integral coincides with the classical Young integral.**

- **Zähle (1998): generalization of these integrals in fractional Sobolev type spaces (also for Hölder spaces).**

- **Pathwise integral:**
  Let $f \in C^{\beta}([0,T];V)$ **and** $\omega \in C^{\beta'}([0,T];V)$**, where** $\alpha < \beta$ **and** $\alpha + \beta' > 1$**. Then**

$$\left| \int_s^t f d\omega \right| \leq c\|f\|_{\beta}\|\omega\|_{\beta'}(t-s)^{\beta'}.$$

- **Nualart & Rascanu (2002); Maslowski & Nualart (2003); G-A, Lu & Schmalfuß (2010); Chen, Gao, G-A, Schmalfuß (2014), Nguyen Dinh Cong, Luu Hoang Duc, Phan Thanh Hong..... Assumption:** $G \in C_b^2$.

- **SPDE's with noise** $\omega \in C^{\beta'}([0,T];V)$, $\beta' \in (1/2,1)$ **generate a random dynamical system.**

Fractional Calculus: case $H \in (1/3, 1/2]$.

We consider the system

$$du(t) = F(u(t))dt + G(u(t))d\omega(t), \qquad u(0) = u_0,$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ **and** $G : \mathbb{R}^d \to L(\mathbb{R}^m, \mathbb{R}^d)$ **are appropriate functions and** ω **is a** β**–Hölder–continuous noisy input with** $\beta \in (1/3, 1/2)$**, considered from** $\mathbb{R}^+$ **to** $\mathbb{R}^m$**.**

We look for u **such that**

$$u(t) = u_0 + \int_0^t F(u(r))dr + \int_0^t G(u(r))d\omega, \quad t \in [0, T].$$

Remember: when $\beta \in (1/2, 1)$**, if** $\alpha + \beta > 1$ **and** $\beta > \alpha$**,**

$$\left| \int_s^t G(u(r))d\omega \right| \leq c\|\omega\|_{\beta,0,T}(t-s)^\beta(1 + \|u\|_\beta).$$

Here $\beta \not> \alpha$**!!!!!!, hence** $D^\alpha_{s+}G(u(\cdot))[r]$ **is not well-defined. Nevertheless, we can provide an explicit formula for**

$$\int_0^t G(u(r))d\omega$$

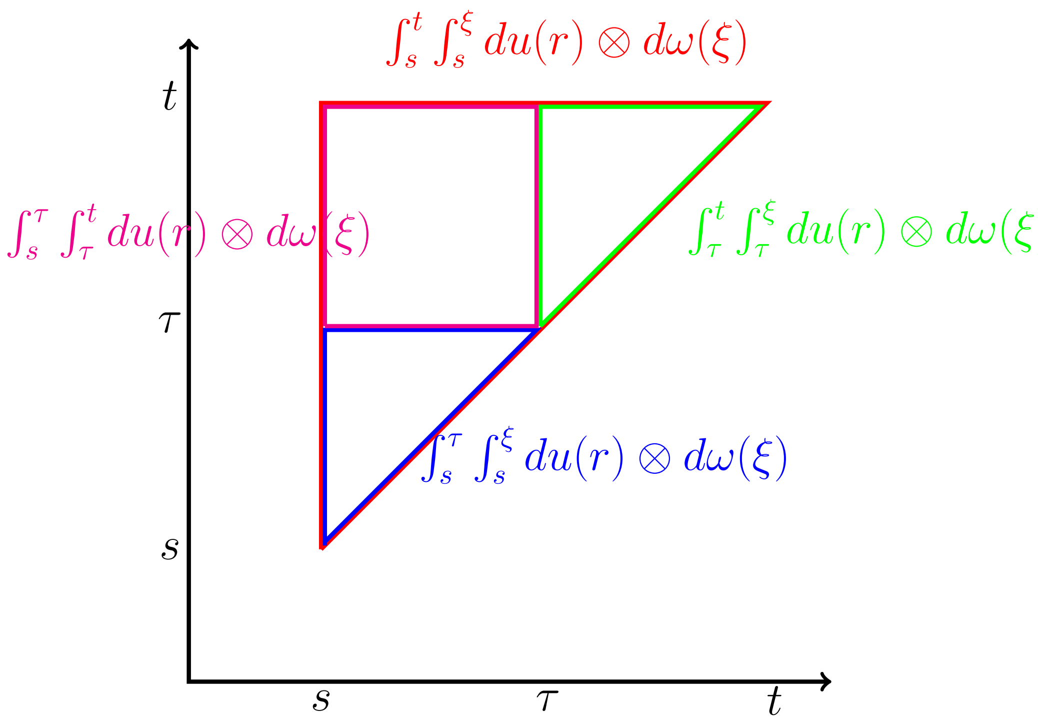
which involves u**,** ω **and the tensor** $u \otimes \omega$**.**

Hu and Nualart (2009): $G \in C_b^3(\mathbb{R}^d)$, $u \in C^\beta([0,T];\mathbb{R}^d)$, $\omega \in C^\beta([0,T];\mathbb{R}^d)$, $1/3 \le \beta < 1/2$, $1-\beta < \alpha < 2\beta$, $\alpha < \frac{\beta+1}{2}$, **such that**

$$(u \otimes \omega)(s,r) + (u \otimes \omega)(r,t) + (u(r) - u(s)) \otimes (\omega(t) - \omega(r)) = (u \otimes \omega)(s,t).$$

For smooth ω, $(s,r) \in \Delta_{0,T} = \{(s,r) : 0 \le s \le r \le T\}$,

$$(u \otimes \omega)(s,r) = \int_s^r (u(\tau) - u(s)) \otimes d\omega(\tau) \in C^{2\beta}(\Delta_{0,T}, \mathbb{R}^d \otimes \mathbb{R}^d).$$



$D^{\alpha}_{s+}G(u(\cdot))[r]$ is not well-defined. However, the expression

$$\hat{D}^{\alpha}_{s+}G(u(\cdot))[r] = \frac{1}{\Gamma(1-\alpha)}\Bigg(\frac{G(u(r))}{(r-s)^{\alpha}} + \alpha\int_s^r \frac{\overbrace{G(u(r)) - G(u(q)) - DG(u(q))(u(r)-u(q))}^{\sim\frac{1}{2}D^2G(u(q))(u(r)-u(q))^2 \sim \|u\|^2_{\beta}(r-q)^{2\beta}}}{(r-q)^{1+\alpha}}dr\Bigg)$$

is well-defined, and

$$\int_0^T G(u(r))d\omega = (-1)^{\alpha}\int_0^T \hat{D}^{\alpha}_{0+}G(u(\cdot))[r]D^{1-\alpha}_{T-}\omega_{T-}[r]dr - (-1)^{2\alpha-1}\int_0^T D^{2\alpha-1}_{0+}(DG(u(\cdot)))[r]D^{1-\alpha}_{T-}\mathcal{D}^{1-\alpha}_{T-}(u\otimes\omega)[r]dr,$$

where

$$\mathcal{D}^{1-\alpha}_{T-}(u\otimes\omega)[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\Bigg(\frac{(u\otimes\omega)(r,T)}{(T-r)^{1-\alpha}} + (1-\alpha)\int_r^T \frac{(u\otimes\omega)(r,q)}{(q-r)^{2-\alpha}}dq\Bigg).$$

The path-area equation

To solve

$$u(t) = u_0 + \int_0^t F(u(r))dr + \int_0^t G(u(r))d\omega$$

we can solve the system (u, v) **with path component**

$$u(t) = u_0 + \int_0^t F(u(r))dr + (-1)^\alpha \int_s^t \hat{D}^\alpha_{s+} G(u(\cdot))[r] D^{1-\alpha}_{t-} \omega_{t-}[r]dr$$
$$- (-1)^{2\alpha-1} \int_s^t D^{2\alpha-1}_{s+} DG(u(\cdot))[r] D^{1-\alpha}_{t-} \mathcal{D}^{1-\alpha}_{t-} v[r]dr,$$

and with area component defined by

$$v(s,t) = \int_s^t \int_s^r F(u(q))dq \otimes d\omega(r)$$
$$+ (-1)^\alpha \int_s^t \hat{D}^\alpha_{s+} G(u(\cdot))[r] D^{1-\alpha}_{t-} (\omega \otimes \omega)(\cdot, t)[r]dr$$
$$- (-1)^{2\alpha-1} \int_s^t D^{2\alpha-1}_{s+} DG(u(\cdot))[r] D^{1-\alpha}_{t-} \mathcal{D}^{1-\alpha}_{t-} (u \otimes (\omega \otimes \omega)(t))(\cdot, t)[r]dr.$$

Then the triplet (u, ω, v) **satisfies the Chen equality**

$$v(s,\tau) + v(\tau, t) + (u(\tau) - u(s)) \otimes (\omega(t) - \omega(\tau)) = v(s,t).$$

The path-area equation

- **Phase space:** $W = W_{0,T} = W_{0,T}(\omega)$ **consisting of pairs**

$$U := (u, v) \in C^{\beta}([0,T]; \mathbb{R}^d) \times C^{2\beta}(\Delta_{0,T}; \mathbb{R}^d \otimes \mathbb{R}^m)$$

such that Chen's relation holds, and we equip this space with the norm

$$\|U\|_W = \|u\|_{\beta,0,T} + \|v\|_{2\beta,\Delta_{0,T}}.$$

- **Theorem: Assume that** $F \in C_b^1(\mathbb{R}^d)$**,** $G \in C_b^3(\mathbb{R}^d)$**,** $\omega \in C^{\beta}([0,T]; \mathbb{R}^d)$**,** $1/3 < \beta < 1/2$**. Then there exists a unique solution** $U = (u, v) \in W$ **on any** $[0,T] \times \Delta_{0,T}$**. Moreover,**

$$\|U\|_W \leq C(\omega)\|u_0\| + \tilde{C}(\omega)(\|F\|_{C_b^1} + \|G\|_{C_b^2})T^{\beta}(1 + \|U\|_W^2), \tag{1}$$

where

$$C(\omega) := c(1 + \|\omega\|_{\beta,0,T}),$$
$$\tilde{C}(\omega) := c(1 + \|\omega\|_{\beta,0,T} + \|\omega\|_{\beta,0,T}^2 + \|(\omega \otimes \omega)\|_{2\beta,\Delta_{0,T}}).$$

- **The** u **component of these solutions generates a random dynamical system** $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$**.**

- **Can be extended to stochastic PDEs (G-A, Lu and Schmalfuß, 2015, 2016).**

Rough Paths

- **We can solve dynamical systems driven by a β-Hölder continuous signal when $\beta \in (\frac{1}{3}, \frac{1}{2})$ using techniques of fractional calculus.**
  But the rough differential equation is transformed into a complicated system of equations involving u, v and $u \otimes (\omega \otimes \omega)$.
- **We will solve m-dimensional dynamical systems**

$$dY_t = f(Y_t)dX_t, \quad Y_0 = \xi,$$

driven by a d-dimensional control function X, based on the fact that we can define the integral

$$\int_0^t Y_s dX_s$$

assuming that Y is controlled by X in the Gubinelli's sense

$$Y_t - Y_s = \mathcal{Y}_s(X_t - X_s) + R_{st}^Y,$$

and using compensated fractional derivatives of Hu and Nualart.

Y. Itô (dissertation, 2015), G.-A., Nualart and Schmalfuß (in preparation).

A β-Hölder continuous rough path is an element

$$(X, \mathbb{X}) \in \mathcal{C}^\beta([0,T];\mathbb{R}^d) := C^\beta([0,T];\mathbb{R}^d) \times C^{2\beta}(\Delta[0,T];\mathbb{R}^{d\times d})$$

that satisfies for any $0 \le r \le \theta \le t \le T$ the Chen's relation

$$\mathbb{X}_{rt} = \mathbb{X}_{r\theta} + \mathbb{X}_{\theta t} + X_{r\theta} \otimes X_{\theta t}.$$

For Y defined on the simplex $\Delta_{0,T}$ and $\gamma > 0$, denote

$$(\Delta_\gamma Y)_t = \int_0^t \frac{|Y_{st}|}{(t-s)^\gamma} ds,$$

where $t \in [0,T]$ (note that if Y is defined on $[0,T]$, $Y_{st} = Y_t - Y_s$).
We denote by $\mathcal{R}_\gamma = \mathcal{R}_\gamma(\mathbb{R}^m)$ the set of measurable functions $R : \Delta_{0,T} \mapsto \mathbb{R}^m$ such that

$$\|R\|_{\mathcal{R}_\gamma} := \sup_{t\in[0,T]} (\Delta_\gamma R)_t + \sup_{(r,t)\in\Delta[0,T]} |R_{rt}| < \infty,$$

and by $\mathcal{Y}_\gamma = \mathcal{Y}_\gamma(\mathbb{R}^m)$ the set of functions $Y : [0,T] \mapsto \mathbb{R}^m$ such that

$$\|Y\|_{\mathcal{Y}_\gamma} := \sup_{t\in[0,T]} ((\Delta_\gamma Y)_t + |Y_t|) = \sup_{t\in[0,T]} (\Delta_\gamma Y)_t + \|Y\|_\infty < \infty.$$

The sets $\mathcal{R}_\gamma$ and $\mathcal{Y}_\gamma$ are Banach spaces.

Fix $\alpha \in (0,1)$ such that

$$1-\beta < \alpha < 2\beta \quad \text{and} \quad 2\alpha < 1+\beta.$$

It is easy to see that, under these constraints we have

$$C^{\beta}([0,T];\mathbb{R}^d) \subset \mathcal{Y}_{2\alpha}(\mathbb{R}^d), \quad C^{2\beta}(\Delta_{0,T};\mathbb{R}^{d\times d}) \subset \mathcal{R}_{\alpha+1}(\mathbb{R}^{d\times d}).$$

Let $X \in C^{\beta}([0,T];\mathbb{R}^d)$. A function $Y:[0,T]\to\mathbb{R}^m$ is controlled by X if there exist $\mathcal{Y} \in \mathcal{Y}_{2\alpha}(\mathbb{R}^{m\times d})$ and $R^Y \in \mathcal{R}_{\alpha+1}(\mathbb{R}^m)$ such that

$$Y_{rt} = \mathcal{Y}_r X_{rt} + R^Y_{rt}, \quad (r,t) \in \Delta_{0,T}.$$

Here $(Y,\mathcal{Y}) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^m)$ is called a controlled rough path, while $\mathcal{Y}$ is the Gubinelli's derivative of Y.

Given $(Y,\mathcal{Y}) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^m)$, one can define

$$\big\|(Y,\mathcal{Y})\big\|_{\mathcal{D}_{X,\alpha}} := \|\mathcal{Y}\|_{\mathcal{Y}_{2\alpha}} + \|R^Y\|_{\mathcal{R}_{\alpha+1}} + |Y_0| + |\mathcal{Y}_0|.$$

Then $\|\cdot\|_{\mathcal{D}_{X,\alpha}}$ is a norm in $\mathcal{D}_{X,\alpha}(\mathbb{R}^m)$.

Lemma Consider a rough path $(X, \mathbb{X}) \in \mathcal{C}^\beta([0,T];\mathbb{R}^d)$ and a controlled path $(Y, \mathcal{Y}) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^m)$. Then, for any $0 \leq s \leq t \leq T$, the integral

$$\begin{aligned} Z_{st} := \int_s^t Y_r dX_r \;&=\; (-1)^\alpha \int_s^t (\hat{D}^\alpha_{s+} Y)_r (D^{1-\alpha}_{t-} X_{t-})_r dr \\ &\quad -(-1)^{2\alpha-1} \int_s^t (D^{2\alpha-1}_{s+} \mathcal{Y})_r (D^{1-\alpha}_{t-} \mathcal{D}^{1-\alpha}_{t-} \mathbb{X})_r dr, \end{aligned}$$

is well-defined. Moreover,

$$\begin{aligned} \left| \int_s^t Y_\theta dX_\theta \right| &\leq c \|X\|_\beta \left((t-s)^\beta \|Y\|_{\mathcal{Y}_{2\alpha}} + (t-s)^{\alpha+\beta} \|R^Y\|_{\mathcal{R}_{\alpha+1}} \right) \\ &\quad + c(\|\mathbb{X}\|_{2\beta} + \|X\|^2_\beta) \|\mathcal{Y}\|_{\mathcal{Y}_{2\alpha}} \left((t-s)^{2\beta} + (t-s)^{2\alpha+2\beta-1} \right), \end{aligned}$$

where $c > 0$ only depends on α and β.

But here

$$\begin{aligned} (\hat{D}^\alpha_{s+} Y)_r \;&=\; \frac{1}{\Gamma(1-\alpha)} \left(\frac{Y_r}{(r-s)^\alpha} + \alpha \int_s^r \frac{Y_{qr} - \mathcal{Y}_q X_{qr}}{(r-q)^{1+\alpha}} dq \right) \\ &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{Y_r}{(r-s)^\alpha} + \alpha \int_s^r \frac{R^Y_{qr}}{(r-q)^{1+\alpha}} dq \right). \end{aligned}$$

Theorem Consider $(X,\mathbb{X}) \in \mathcal{C}^{\beta}([0,T];\mathbb{R}^d)$, $\beta \in (1/3, 1/2]$ and let $(Y,\mathcal{Y}) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^{m\times d})$. Let $Z_t := Z_{0t}$ be the process defined above. Then $(Z,Y) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^m)$, that is,

$$Z_{st} = Y_s X_{st} + R^Z_{st}$$

with the residual R^Z given by

$$R^Z_{st} = (-1)^{\alpha} \int_s^t (\widehat{D}^{\alpha}_{s+} Y_{s+})_r (D^{1-\alpha}_{t-} X_{t-})_r dr - (-1)^{2\alpha-1} \int_s^t (D^{2\alpha-1}_{s+} \mathcal{Y})_r (D^{1-\alpha}_{t-} \mathcal{D}^{1-\alpha}_{t-} \mathbb{X})_r dr,$$

where $(Y_{s+})_r = Y_{sr}$.

Stability of $\mathcal{D}_{X,\alpha}(\mathbb{R}^m)$ under a smooth path: **We show that the composition of a controlled path with a regular function is still a controlled path.**

Lemma **Let** $f \in C_b^2(\mathbb{R}^m; \mathbb{R}^{m\times d}))$. **Consider** $(X, \mathbb{X}) \in \mathcal{C}^\beta([0,T];\mathbb{R}^d)$ **for some** $\beta \in (1/3, 1/2]$ **and let** $(Y, \mathcal{Y}) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^m)$. **Then** $(f(Y), \mathcal{Y}^{f(Y)}) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^{m\times d})$, **with residual term** $R^{f(Y)}$, **where**

$$\mathcal{Y}_t^{f(Y)} := f'(Y_t)\mathcal{Y}_t, \quad R_{st}^{f(Y)} := f(Y_t) - f(Y_s) - f'(Y_s)\mathcal{Y}_s X_{st}.$$

General definition of the integral:

Lemma **Assume** $f \in C_b^2(\mathbb{R}^m; \mathbb{R}^{m\times d})$, $(X, \mathbb{X}) \in \mathcal{C}^\beta([0,T];\mathbb{R}^d)$. **Let** $(Y, \mathcal{Y}) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^m)$. **Then the integral**

$$\begin{aligned}\mathcal{Z}_{st} := \int_s^t f(Y_r)dX_r = (-1)^\alpha \int_s^t (\hat{D}_{s+}^\alpha f(Y))_\tau (D_{t-}^{1-\alpha} X_{t-})_\tau d\tau \\ - (-1)^{2\alpha-1} \int_s^t (D_{s+}^{2\alpha-1}\mathcal{Y}^{f(Y)})_\tau (D_{t-}^{1-\alpha}\mathcal{D}_{t-}^{1-\alpha}\mathbb{X})_\tau d\tau,\end{aligned}$$

is well-defined.

Theorem Assume $\xi \in \mathbb{R}^m$**,** $f \in C_b^3(\mathbb{R}^m; \mathbb{R}^{m\times d})$**,** $(X, \mathbb{X}) \in \mathcal{C}^\beta([0,T]; \mathbb{R}^d)$**. Then for any** $t \in [0,T]$ **there exists a unique** $(Y, \mathcal{Y}) \in \mathcal{D}_{X,\alpha}(\mathbb{R}^m)$ **solution of**

$$Y_t = \xi + \int_0^t f(Y_r)dX_r =: \xi + \mathcal{Z}_{0t}, \quad t \geq 0.$$

Sketch of the proof. We apply a fixed point theorem applied to the mapping $\Phi(Y, \mathcal{Y}) : \mathcal{D}_{X,\alpha}(\mathbb{R}^m) \to \mathcal{D}_{X,\alpha}(\mathbb{R}^m)$ **defined by**

$$\Phi(Y, \mathcal{Y}) := \left(\xi + \int_0^\cdot f(Y_r)dX_r, f(Y)\right).$$

Approach of Nualart and Răşcanu, defining for any $\alpha, \lambda \geq 0$ **and for any function** Y **on the simplex,**

$$\|Y\|_{\lambda,\alpha} := \sup_{t\in[0,T]} e^{-\lambda t}(\Delta_\alpha Y)_t = \sup_{t\in[0,T]} e^{-\lambda t} \int_0^t \frac{|Y_{st}|}{(t-s)^\alpha} ds.$$

If Y **is a function on** $[0,T]$**, we set**

$$\|Y\|_\lambda = \sup_{t\in[0,T]} e^{-\lambda t}|Y_t|.$$

We introduce the following seminorm in the space $\mathcal{D}_{X,\alpha}(\mathbb{R}^m)$**:**

$$\|(Y, \mathcal{Y})\|_{\lambda,\alpha} = \|Y\|_\lambda + \|\mathcal{Y}\|_\lambda + \|Y\|_{\lambda,2\alpha} + \|\mathcal{Y}\|_{\lambda,2\alpha} + \|R^Y\|_{\lambda,\alpha+1}.$$

Invariance: **There exists λ_0 such that the set**

$$\mathcal{B}_{\lambda_0} = \{(Y, \mathcal{Y}) \in \mathcal{D}_{X,\alpha} : \|\mathcal{Y}\|_\infty \leq \|f\|_\infty, \|(Y, \mathcal{Y})\|_{\lambda_0,\alpha} \leq |\xi| + \|f\|_\infty + 1\}$$

is invariant under the mapping Φ. That is,

$$\|(Y, \mathcal{Y})\|_{\lambda_0,\alpha} \leq |\xi| + \|f\|_\infty + 1 \Rightarrow \|\Phi(Y, \mathcal{Y})\|_{\lambda_0,\alpha} \leq |\xi| + \|f\|_\infty + 1.$$

Contraction: **There exists $\lambda_1 \geq \lambda_0$ such that for any $(Y^1, \mathcal{Y}^1)$ and $(Y^2, \mathcal{Y}^2)$ such that $\|\mathcal{Y}^i\|_\infty \leq \|f\|_\infty$, $\|(Y^i, \mathcal{Y}^i)\|_{\lambda_0,\alpha} \leq |\xi| + \|f\|_\infty + 1$, we have**

$$\|(\Phi(Y^1, \mathcal{Y}^1) - \Phi(Y^2, \mathcal{Y}^2))\|_{\lambda_1,\alpha} \leq \frac{1}{2}\|(Y^1, \mathcal{Y}^1) - (Y^2, \mathcal{Y}^2))\|_{\lambda_1,\alpha}.$$

If

$$Y_t = \int_0^t f(Y_r)dX_r, \quad t \geq 0,$$

then we can prove the additivity of the integral

$$\int_0^t f(Y_r)dX_r = \int_0^s f(Y_r)dX_r, + \int_s^t f(Y_r)dX_r.$$

Still open in the general case!

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THANK YOU

Local exponential stability of the trivial solution

Assume

$$F(0) = 0, \quad G(0) = 0.$$

Denote by $\varphi(t, \omega, u_0)$ the first component of the solution.

Definition: The trivial solution of the above problem is said to be exponential stable with rate $\mu > 0$ if there exists a random variable $\alpha(\omega) > 0$ and a random neighborhood $\mathcal{U}_0(\omega)$ of zero such that for all $\omega \in \Omega$ and $t \in \mathbb{R}^+$

$$\sup_{u_0 \in \mathcal{U}_0(\omega)} \|\varphi(t, \omega, u_0)\| \leq \alpha(\omega) e^{-\mu t}.$$

The method consists on

1. **Cut–off strategy: $(u^n, v^n)_{n\in\mathbb{N}}$ solution of modified system depending on random variables.**
2. **Discrete Gronwall-like lemma: subexponential estimates of $(u^n)_{n\in\mathbb{N}}$.**
3. **Temperedness of random variables.**

A random variable $R \in (0, \infty)$ is called tempered from above if

$$\limsup_{t \to \pm\infty} \frac{\log^+ R(\theta_t \omega)}{t} = 0 \qquad \text{for almost all} \quad \omega \in \Omega.$$

Example. $\|\omega\|_\beta$ **is tempered from above.**

R **is called tempered from below if** R^{-1} **is tempered from above. Then for any** $\epsilon > 0$ **there exists a (random) constant** $C_\epsilon(\omega) > 0$ **such that**

$$R(\theta_t \omega) \geq C_\epsilon(\omega) e^{-\epsilon|t|} \qquad \text{for almost all} \quad \omega \in \Omega.$$

We assume

$$f(u) := F(u) - Au$$

where A **is a negative definite linear operator that generates the fundamental solution** e^{At} **to the linear equation** $du(t) = Au(t)dt$**. Then** $f \in C_b^1(\mathbb{R}^d)$ **and** $f(0) = 0$**.**

Assume that $\operatorname{Re} \sigma(A) < -\lambda < 0$**. Then there exists** $M \geq 1$ **such that**

$$\|e^{At}\| \leq M e^{-\lambda t}.$$

Consider

$$du(t) = (Au(t) + f(u(t)))dt + G(u(t))d\omega(t), \qquad u(0) = u_0$$

Consider the mild version of this equation

$$\begin{aligned}
u(t) &= e^{At}u_0 + \int_0^t e^{A(t-r)} f(u(r))dr \\
&\quad + (-1)^\alpha \int_0^t \hat{D}_+^\alpha e^{A(t-\cdot)} G(u(\cdot))[r] D_{t-}^{1-\alpha}\omega_{t-}[r]dr \\
&\quad - (-1)^{2\alpha-1} \int_0^t D_{0+}^{2\alpha-1} e^{A(t-\cdot)} DG(u(\cdot))[r] D_{t-}^{1-\alpha}\mathcal{D}_{t-}^{1-\alpha} v[r]dr,
\end{aligned}$$

$$\begin{aligned}
v(s,t) &= \int_s^t \int_s^r (Au(q) + f(u(q)))dq \otimes d\omega(r) \\
&\quad + (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha}(\omega \otimes \omega)(\cdot, t)[r]dr \\
&\quad - (-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} DG(u(\cdot))[r] D_{t-}^{1-\alpha}\mathcal{D}_{t-}^{1-\alpha}(u \otimes (\omega \otimes \omega)(t))(\cdot, t)[r]dr.
\end{aligned}$$

Then the solution is equal to the solution of the original equation.

Cut-off of the modified path-area equation

We apply a cut-off technique:

$$\chi : \mathbb{R}^d \to \bar{B}_{\mathbb{R}^d}(0,1), \qquad \chi(u) = \begin{cases} u & : \quad \|u\| \le \frac{1}{2} \\ 0 & : \quad \|u\| \ge 1 \end{cases}$$

where $D\chi$ and $D^2\chi$ are bounded by $L_{D\chi}$, $L_{D^2\chi}$.

$$\chi_{\hat{R}(\omega)}(u) = \hat{R}(\omega)\chi\left(\frac{u}{\hat{R}(\omega)}\right) \in \bar{B}_{\mathbb{R}^d}(0, \hat{R}(\omega)),$$

$$f_{\hat{R}(\omega)} := f \circ \chi_{\hat{R}(\omega)} : \mathbb{R}^d \to \mathbb{R}^d, \qquad G_{\hat{R}(\omega)} := G \circ \chi_{\hat{R}(\omega)} : \mathbb{R}^d \to L(\mathbb{R}^m, \mathbb{R}^d)$$

Lemma. Assume that $Df(0) = 0$, $DG(0) = 0$ and $D^2G(0) = 0$. Then, for every $R > 0$ there exists a positive $\hat{R}$ such that for u, $z \in \mathbb{R}^d$

$$\begin{aligned}
&\|f_{\hat{R}}(u)\| \le L_{D\chi} R \|u\|, \\
&\|G_{\hat{R}}(u)\| \le L_{D\chi} R \|u\|, \qquad \|DG_{\hat{R}}(u)\| \le L_{D\chi}^2 R \|u\|, \\
&\|G_{\hat{R}}(u) - G_{\hat{R}}(z)\| \le L_{D\chi} R \|u - z\|, \\
&\|DG_{\hat{R}}(u) - DG_{\hat{R}}(z)\| \le (L_{D\chi}^2 + L_{D^2\chi}) R \|u - z\|, \\
&\|G_{\hat{R}}(u) - G_{\hat{R}}(z) - DG(\chi_{\hat{R}}(z))(\chi_{\hat{R}}(u) - \chi_{\hat{R}}(z))\| \le (L_{D\chi}^2 + L_{D^2\chi}) R \|u - z\|^2.
\end{aligned}$$

Lemma: For (small and tempered from below) $R(\omega)$ we find a (tempered from below) $\hat{R}(\omega)$ such that

$$\left\| \int_0^\cdot e^{A(\cdot-r)} f_{\hat{R}(\omega)}(u(r))dr \right\|_{\beta,0,1} + \left\| \int_0^\cdot e^{A(\cdot-r)} G_{\hat{R}(\omega)}(u(r))d\omega(r) \right\|_{\beta,0,1}$$
$$\leq C_1(\omega)R(\omega)\|u\|_{\beta,0,1}(1+\|u\|_{\beta,0,1}+\|v\|_{2\beta,\Delta_{0,1}}),$$

where

$$C_1(\omega) := cM^2(1+\|A\|)^2 \max\{L_{D\chi}, L_{D\chi}^2 + L_{D^2\chi}\}(1+|\!|\!|\omega|\!|\!|_{\beta,0,1}),$$

hence $C_1(\omega)$ is tempered from above.

Corollary: If $U=(u,v)\in W_{0,1}(\omega)$ is the path–area solution corresponding to the nonlinear functions $f_{\hat{R}(\omega)}$ and $G_{\hat{R}(\omega)}$, then

$$\|u\|_{\beta,0,1} \leq M(1+\|A\|)\|u_0\| + C_1(\omega)R(\omega)\|u\|_{\beta,0,1}(1+\|u\|_{\beta,0,1}+\|v\|_{2\beta,\Delta_{0,1}}).$$

If $U=(u,v)\in W_{0,1}(\omega)$ is the path–area solution corresponding to the nonlinear functions $f_{\hat{R}(\omega)}$ and $G_{\hat{R}(\omega)}$, then

$$\|v\|_{2\beta,\Delta_{0,1}} \leq C_2(\omega)\|u_0\| + C_3(\omega)R(\omega)\|u\|_{\beta,0,1}(1+\|u\|_{\beta,0,1}+\|v\|_{2\beta,\Delta_{0,1}})$$

with

$$C_2(\omega) \sim |\!|\!|\omega|\!|\!|_{\beta,0,T}\,, \qquad C_3(\omega) \sim |\!|\!|\omega|\!|\!|_{\beta,0,T}\,, \|(\omega\otimes\omega)\|_{2\beta,\Delta_{0,T}}.$$

$$\|U\|_W \leq K_1(\omega)\|u_0\| + K_2(\omega)R(\omega)(1 + \|U\|_W^2),$$

where K_1 and K_2 are positive tempered from above random variables:

$$K_1(\omega),\ K_2(\omega) \sim \left\|\!\left|\omega\right|\!\right\|_{\beta,0,1},\ \|(\omega \otimes \omega)\|_{2\beta,\Delta_{0,1}}.$$

Assume that $R(\omega)$ and u_0 are chosen such that

$$4\left(K_1(\omega)\|u_0\| + K_2(\omega)R(\omega)\right)K_2(\omega)R(\omega) < 1,$$

Then

$$\|U\|_W \leq 2(K_1(\omega)\|u_0\| + K_2(\omega)R(\omega)).$$

In fact, consider $y = ay^2 + b$ with $a = K_2(\omega)R(\omega)$ and $b = K_1(\omega)\|u_0\| + K_2(\omega)R(\omega)$. We then have

$$\|U\|_W \leq y_1 \leq \frac{1 - \sqrt{1 - 4ab}}{2a} = \frac{1 - (1 - 4ab)}{2a(1 + \sqrt{1 - 4ab})} \leq 2b.$$

Moreover, $2b \leq 1$ if $\|u_0\|$ is sufficiently small and $R(\omega)$ too. For $\epsilon \in (0,1)$ assume:

$$3K_2(\omega)R(\omega) = \epsilon,$$

hence $R(\omega)$ is tempered from below. Consider $\rho_0(\omega)$ such that $u_0 \in B_{\mathbb{R}^d}(0, \rho_0(\omega))$ such that

$$K_1(\omega)\rho_0(\omega) + \frac{\epsilon}{3} \leq \frac{1}{2}.$$

Estimates of the solution

Consider $(U^n)_{n\in\mathbb{N}\cup\{0\}} = ((u^n, v^n))_{n\in\mathbb{N}\cup\{0\}}$ **a sequence of path-area solutions on** $W_{0,1}(\theta_n\omega)$**, where the path component is given for** $t \in [0,1]$ **by**

$$u^n(t) = e^{At}u^n(0) + \int_0^t e^{A(t-r)} f_{\hat{R}(\theta_n\omega)}(u^n(r))dr + \int_0^t e^{A(t-r)} G_{\hat{R}(\theta_n\omega)}(u^n(r))d\theta_n\omega(r),$$

such that $u^{n-1}(1) = u^n(0)$**, being** $u^{-1}(1) = u_0$**.**

Consider the u^0 **component of the solution** U^0 **on** $[0,1]$ **for** ω**. We can assume that** $\|U^0\|_W \leq 1$ **for** u_0 **such that** $\|u_0\| \leq \rho_0(\omega)$**:**

$$\begin{aligned} \|u^0\|_{\beta,0,1} &\leq M(1+\|A\|)\|u_0\| + C_1(\omega)R(\omega)\|u^0\|_{\beta,0,1}(1+\|u^0\|_{\beta,0,1} + \|v^0\|_{2\beta,\Delta_{0,1}}) \\ &\leq M(1+\|A\|)\|u_0\| + 3C_1(\omega)R(\omega)\|u^0\|_{\beta,0,1} \\ &\leq c_A\|u_0\| + \epsilon\|u^0\|_{\beta,0,1} \end{aligned}$$

which implies

$$\|u^0(1)\| \leq \|u^0\|_{\beta,0,1} \leq \frac{c_A}{1-\epsilon}\|u_0\| := c_{A,\epsilon}\|u_0\|$$

Let $\rho_1(\omega) = \rho_1(\omega, \epsilon) \leq \rho_0(\omega)$ **be the maximal radius such that for** $u_0 \in B_{\mathbb{R}^d}(0, \rho_1(\omega))$ **we have**

$$c_{A,\epsilon} K_1(\theta_1\omega)\rho_1(\omega) + \frac{\epsilon}{3} \leq \frac{1}{2},$$

then

$$\begin{aligned} &4\left(K_1(\theta_1\omega)\|u^0(1)\| + K_2(\theta_1\omega)R(\theta_1\omega)\right) K_2(\theta_1\omega)R(\theta_1\omega) \\ &\leq 4\bigg(c_{A,\epsilon}K_1(\theta_1\omega)\|u_0\| + \frac{\epsilon}{3}\bigg)\frac{\epsilon}{3} < 1. \end{aligned}$$

Since

$$\begin{aligned} u^1(t) = e^{At}\bigg(&e^A u_0 + \int_0^1 e^{A(1-r)} f_{\hat{R}(\omega)}(u^0(r))dr + \int_0^1 e^{A(1-r)} G_{\hat{R}(\omega)}(u^0(r))d\omega\bigg) \\ &+ \int_0^t e^{A(t-r)} f_{\hat{R}(\theta_1\omega)}(u^1(r))dr + \int_0^t e^{A(t-r)} G_{\hat{R}(\theta_1\omega)}(u^1(r))d\theta_1\omega, \end{aligned}$$

then

$$\begin{aligned} \|u^1\|_{\beta,0,1} &\leq M(1 + \|A\|)(Me^{-\lambda}\|u_0\| + \epsilon\|u^0\|_{\beta,0,1}) + \epsilon\|u^1\|_{\beta,0,1} \\ &\leq c_A(e^{-\lambda} + \epsilon c_{A,\epsilon})\|u_0\| + \epsilon\|u^1\|_{\beta,0,1}. \end{aligned}$$

Therefore

$$\|u^1(1)\| \leq \|u^1\|_{\beta,0,1} \leq c_{A,\epsilon}(e^{-\lambda} + \epsilon c_{A,\epsilon})\|u_0\| = c_{A,\epsilon} e^{\log(e^{-\lambda} + \epsilon c_{A,\epsilon})}\|u_0\|$$

Let $\rho_2(\omega) \leq \rho_1(\omega)$ **be the maximal radius such that for** $u_0 \in B_{\mathbb{R}^d}(0, \rho_2(\omega))$

$$c_{A,\epsilon} e^{\log(e^{-\lambda}+\epsilon c_{A,\epsilon})} K_1(\theta_2\omega)\rho_2(\omega) + \frac{\epsilon}{3} \leq \frac{1}{2},$$

then $\|U^2\|_W \leq 1$**. Since for** $t \in [0,1]$

$$\begin{aligned} u^2(t) = e^{At}\Big(& e^{2A}u_0 + e^A \int_0^1 e^{A(1-r)} f_{\hat{R}(\omega)}(u^0(r))dr + e^A \int_0^1 e^{A(1-r)} G_{\hat{R}(\omega)}(u^0(r))d\omega \\ & + \int_0^1 e^{A(1-r)} f_{\hat{R}(\theta_1\omega)}(u^1(r))dr + \int_0^1 e^{A(1-r)} G_{\hat{R}(\theta_1\omega)}(u^1(r))d\theta_1\omega \Big) \\ & + \int_0^t e^{A(t-r)} f_{\hat{R}(\theta_2\omega)}(u^2(r))dr + \int_0^t e^{A(t-r)} G_{\hat{R}(\theta_2\omega)}(u^2(r))d\theta_2\omega, \end{aligned}$$

we obtain

$$\begin{aligned} \|u^2\|_{\beta,0,1} &\leq M(1+\|A\|)(Me^{-2\lambda}\|u_0\| + Me^{-\lambda}\epsilon\|u^0\|_{\beta,0,1} + \epsilon\|u^1\|_{\beta,0,1}) + \epsilon\|u^2\|_{\beta,0,1} \\ &\leq c_A e^{-2\lambda}(\|u_0\| + \epsilon e^{\lambda}\|u^0\|_{\beta,0,1} + \epsilon e^{2\lambda}\|u^1\|_{\beta,0,1}) + \epsilon\|u^2\|_{\beta,0,1}, \end{aligned}$$

hence applying a discrete version of Gronwall's lemma,

$$\|u^2(1)\| \leq \|u^2\|_{\beta,0,1} \leq c_{A,\epsilon} e^{2\log(e^{-\lambda}+\epsilon c_{A,\epsilon})}\|u_0\|.$$

Iteration and Gronwall lemma...for $\rho_n(\omega) \le \rho_{n-1}(\omega)$

$$\|u^n(1)\| \le \|u^n\|_{\beta,0,1} \le e^{-n\lambda} c_{A,\epsilon} \|u_0\| (1 + \epsilon c_{A,\epsilon} e^{\lambda})^n \le c_{A,\epsilon} \|u_0\| e^{n \log(e^{-\lambda} + \epsilon c_{A,\epsilon})}.$$

We have constructed a finite number of elements of a sequence of radii

$$\rho_n(\omega) \le \rho_{n-1}(\omega) \le \cdots \le \rho_0(\omega)$$

Temperedness of K_1: there is a natural number $N(\omega,\epsilon)$ such that for $n \ge N(\omega,\epsilon)$

$$c_{A,\epsilon} e^{(n-1)\log(e^{-\lambda}+\epsilon c_{A,\epsilon})} K_1(\theta_n\omega) \rho_{N(\omega,\epsilon)}(\omega) + \frac{\epsilon}{3} \le \frac{1}{2}.$$

For $n \ge N(\omega,\epsilon)$ we define $\rho_n(\omega) = \rho_n(\omega,\epsilon) := \rho_{N(\omega,\epsilon)}(\omega)$, such that for all $n \in \mathbb{N}$

$$c_{A,\epsilon} e^{(n-1)\log(e^{-\lambda}+\epsilon c_{A,\epsilon})} K_1(\theta_n\omega) \rho_n(\omega) + \frac{\epsilon}{3} \le \frac{1}{2}.$$

$$\|u^n\|_{\beta,0,1} \le c_{A,\epsilon} e^{n\log(e^{-\lambda}+\epsilon c_{A,\epsilon})}\|u_0\|.$$

Relation to the original equation (without cut–off): There is a $\hat\rho(\omega) \le \rho_N(\omega)$ such that for $u_0 \in \mathcal{U}_0 := B_{\mathbb{R}^d}(0,\hat\rho(\omega))$ we have $\hat R$ is tempered from below, then

$$\|u^n\|_{\beta,0,1} \le \frac{\hat R(\theta_n\omega)}{2}, \qquad \textbf{for } \; n \in \mathbb{Z}^+.$$

Hence, as a consequence of the definition of $\chi_{\hat R}$,

$$f_{\hat R(\theta_n\omega)}(u^n(r)) = f(u^n(r)), \qquad G_{\hat R(\theta_n\omega)}(u^n(r)) = G(u^n(r)).$$

Construction of the path area solution:

$$(u(t), v(s,t)) := (u^n(t-n), v^n(s-n, t-n)),\ (t,(s,t)) \in [n, n+1] \times \Delta_{n,n+1}.$$

(u,v) solves the path area solution.
Main tool: path-area concatenation

$$u(t,\omega) = \begin{cases} u^0(t,\omega) & : \ t \in [0,1], \\ u^1(t-1,\omega) & : \ t \in [1,2]. \end{cases}$$

$$v(s,t,\omega) = \begin{cases} v^0(s,t,\omega) & : \quad s \le t \in \Delta_{0,1}, \\ v^1(s-1,t-1,\theta_1\omega) & : \quad s \le t \in \Delta_{1,2}, \\ v^0(s,1,\omega) + v^1(0,t-1,\theta_1\omega) & \\ \quad +(u^0(1,\omega) - u^0(s,\omega)) \otimes (\omega(t) - \omega(1)) & : \ (s,t) \in [0,1] \times [1,2]. \end{cases}$$

Theorem. **There exists a neighborhood $\mathcal{U}_0(\omega)$ of zero such that if u_0 is contained in $\mathcal{U}_0(\omega)$ the path part of the path–area solution is exponentially stable with an exponential rate less than $\mu < \lambda$.**

Proof: **Take $0 < \mu < \lambda$. For $\epsilon \in I := (0, \frac{1-e^{-\lambda}}{c_A+1-e^{-\lambda}})$ we define $\mu(\epsilon) := -\log(e^{-\lambda} + \epsilon c_{A,\epsilon})$. There exists $\epsilon \in I$ such that $\mu(\epsilon) \geq \mu$.**

Consider $u_0 \in \mathcal{U}_0 := B_{\mathbb{R}^d}(0, \hat{\rho}(\omega))$. Given $t \in [n, n+1]$, $n \in \mathbb{N}$, we obtain

$$n \log(e^{-\lambda} + \epsilon c_{A,\epsilon}) = -n\mu(\epsilon) \leq (1-t)\mu(\epsilon),$$

then

$$\|u(t)\| \leq \|u^n\|_{\beta,0,1} \leq c_{A,\epsilon}\|u_0\|e^{\mu(\epsilon)}e^{-\mu(\epsilon)t} \leq c_{A,\epsilon}\hat{\rho}(\omega)e^{\mu(\epsilon)}e^{-\mu t},$$

which leads to the desired local exponential stability

$$\sup_{u_0 \in \mathcal{U}_0(\omega)} \|\varphi(t, \omega, u_0)\| \leq \alpha(\omega)e^{-\mu t},$$

taking

$$\alpha(\omega) = c_{A,\epsilon}\hat{\rho}(\omega)e^{\mu(\epsilon)}.$$